The Sundial Problem from a New Angle

Russell Goyder

NASS Conference 2025

Agenda

Summary of topics to cover in this presentation

Motivation

How I became interested in sundials

Setup and Definitions

Earth's orbit and the location and geometry of the sundial and shadow

The Solution

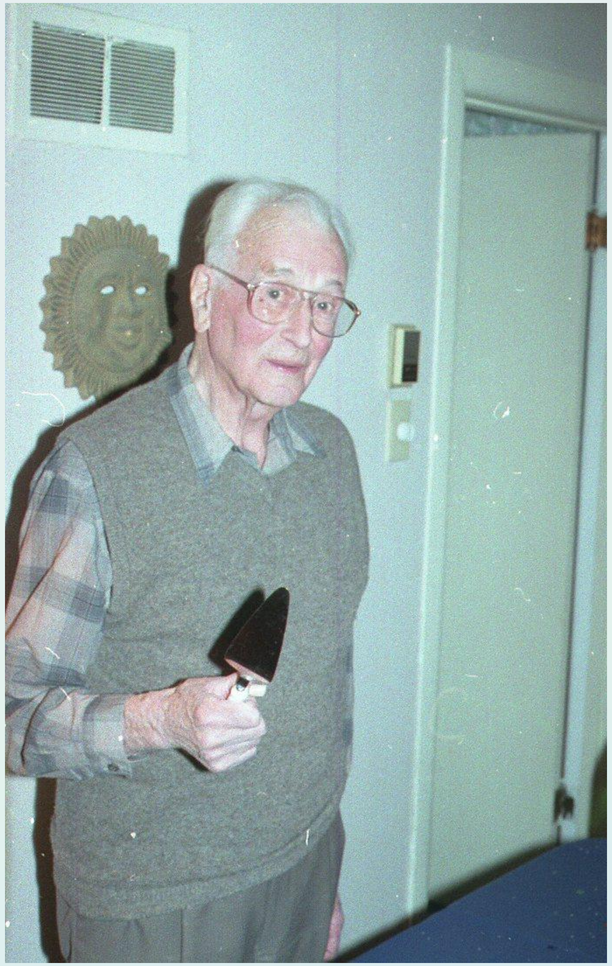
An exact parametric expression for the analemma and Equation of Time

The Literature

Where does this work fit into the existing literature? Is there anything new here?

Motivation

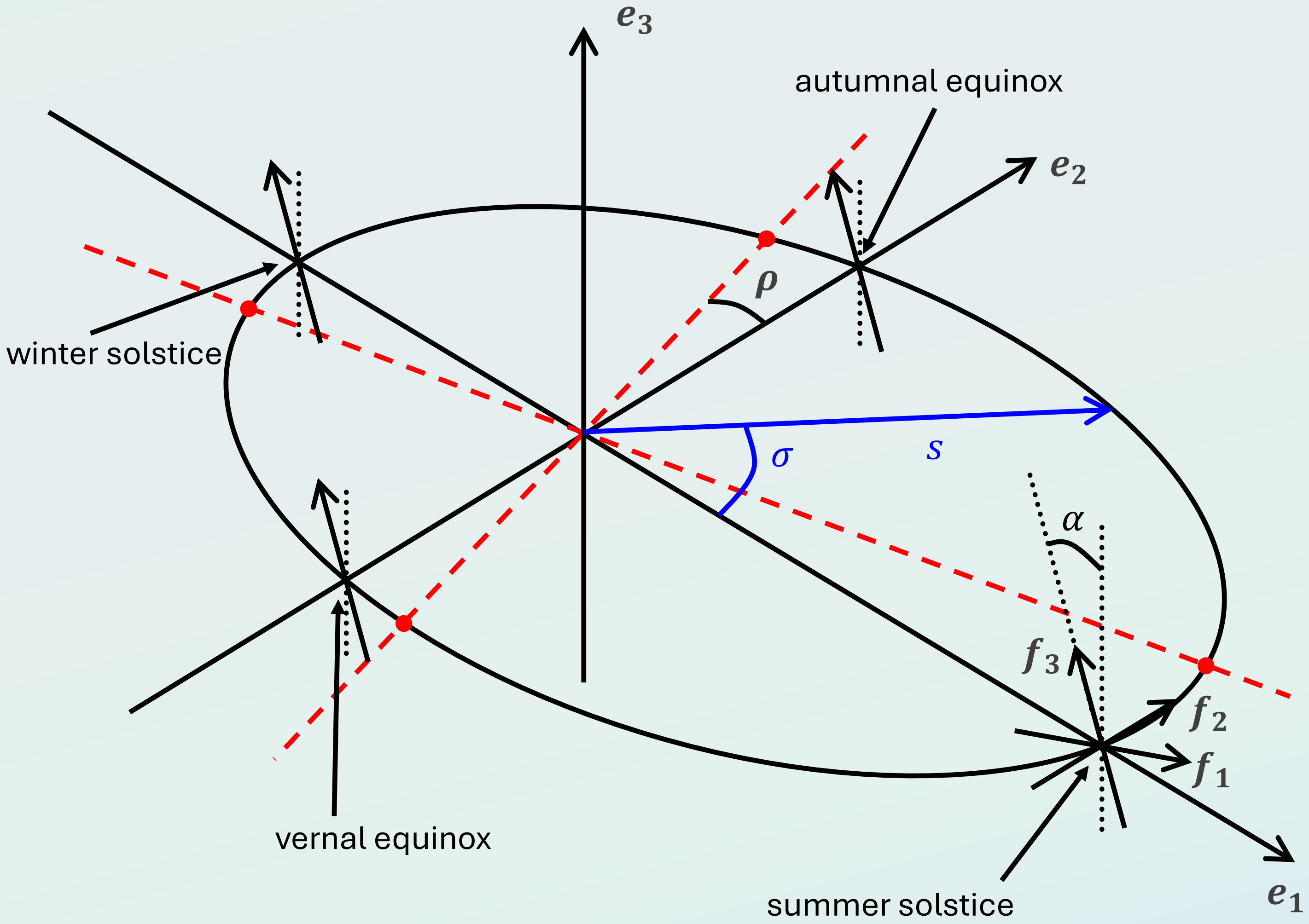
How did I become interested in sundials?



When visiting my grandfather Claude Goyder (1912 – 2007) in Waterloo, Ontario in 2005, he casually estimated the time from a shadow. I thought about the sundial problem on my flight back to the UK.

Earth's Orbit

Defining a coordinate frame for Earth's orbit around the Sun



e_1 and e_2 are in the plane of Earth's orbit around the sun

e_3 is perpendicular to the plane of Earth's orbit

s is a vector from the center of the Sun to the center of the Earth – a sun ray

σ measures the progress of Earth's orbit

ρ measures the offset of the solstices from perigee and apogee

α measures the tilt of Earth's axis of rotation relative to the orbital plane

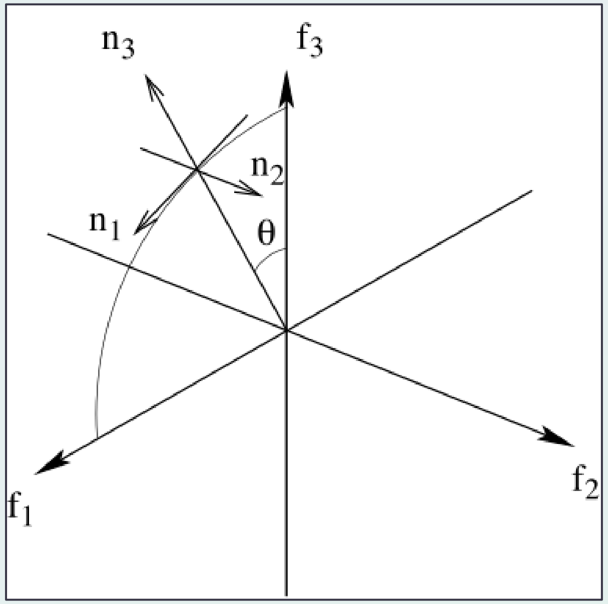
f_3 points along Earth's axis of rotation

f_1 and f_2 are in Earth's equatorial plane

The Sundial

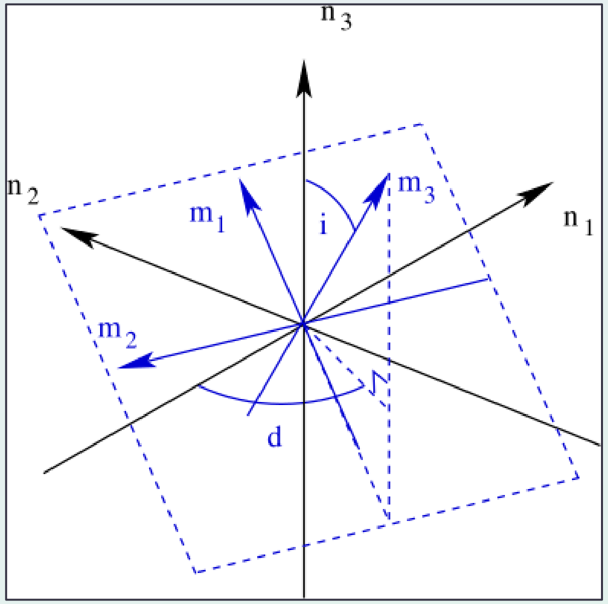
Defining the location and orientation of the sundial and gnomon

Surface frame



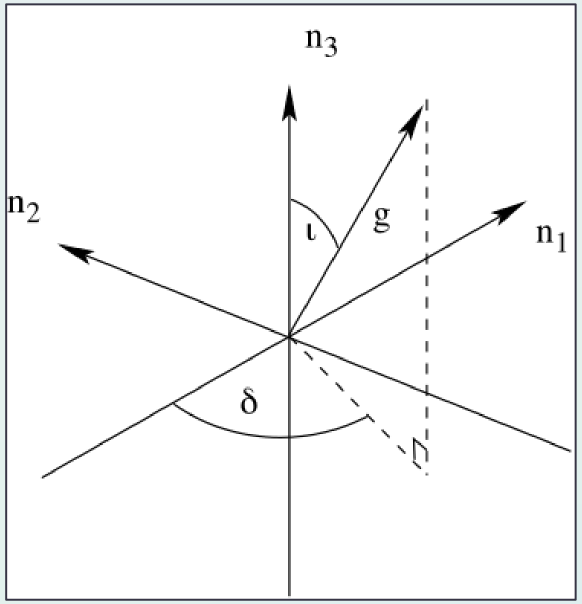
- $\boldsymbol{n_3}$ points overhead
- $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ lie in the ground
- $\boldsymbol{\theta}$ defines (90° minus) latitude

Dial face



- $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ lie in dial face
- $\boldsymbol{i}$ is the dial's inclination
- $\boldsymbol{d}$ is the dial's declination

Gnomon

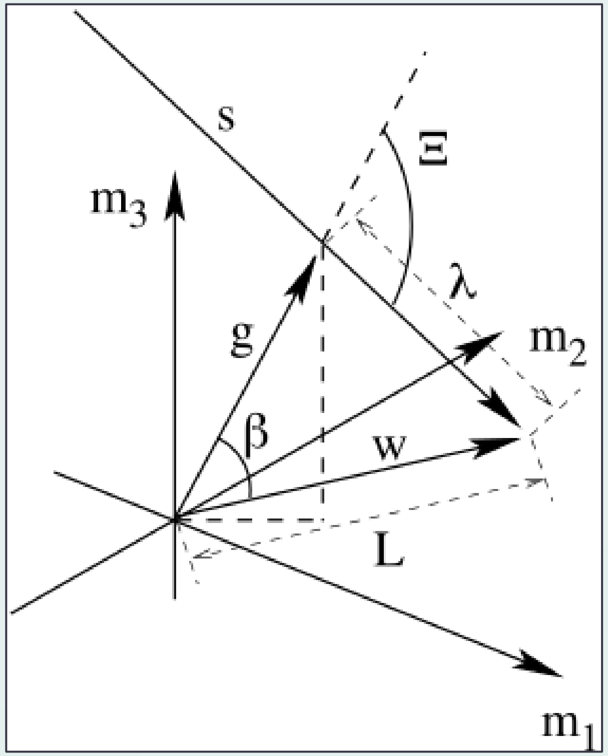


- $\boldsymbol{g}$ is the gnomon
- $\boldsymbol{\iota}$ is the gnomon's inclination
- $\boldsymbol{\delta}$ is the gnomon's declination

The Shadow

Geometry of the shadow cast by the gnomon on the dial face

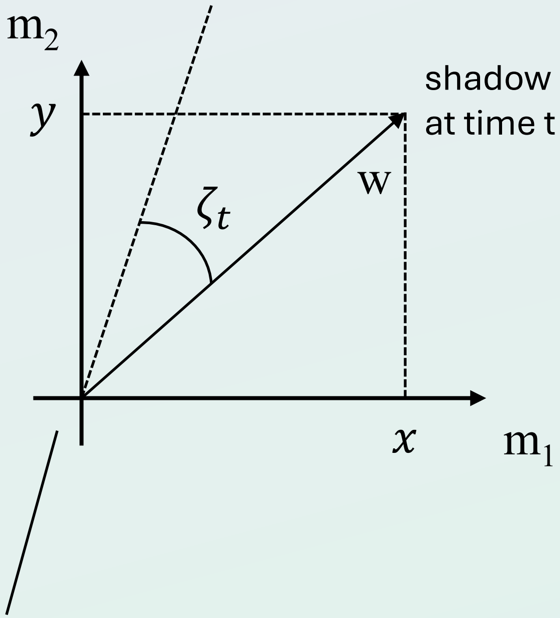
Shadow triangle



- **w** is the shadow, length **L**
- The sun ray **s**, gnomon **g** and shadow **w** all lie in the *shadow plane*

shadow at noon

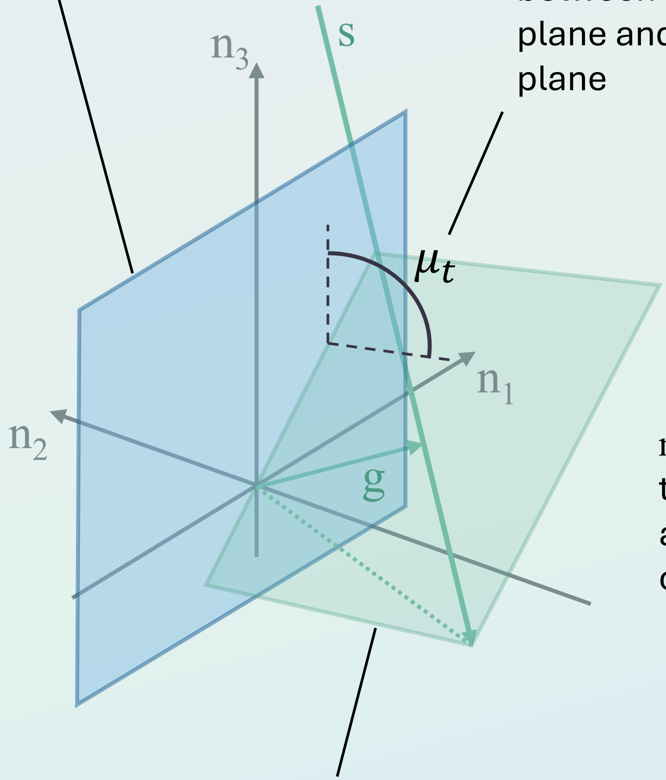
shadow at time t



The dial face containing the vectors m_1 and m_2

The meridian plane containing a line of longitude and the vectors n_1 and n_3

The hour angle μ_t between the shadow plane and the meridian plane



n_1 and n_2 lie in the ground and n_3 points overhead

The shadow plane containing the sun ray s and the gnomon g

The Solution – Geometry

Sundial and shadow geometry in terms of Earth's orbit σ_t and rotation ψ_t

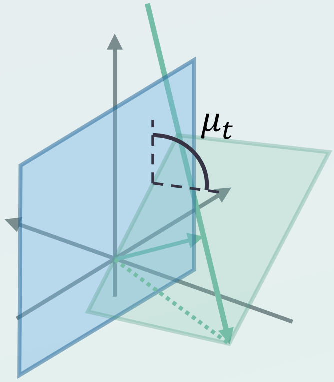
I was able to find a nice expression for $\sin\Xi\cos\mu_t$ in closed form, but I could only find a nice factorization for $\sin\Xi\sin\mu_t$ when $\delta = 0$ (gnomon with zero declination). All results below are given in this special case.

$$\tan\mu_t = \frac{\sin\psi_t\cos\sigma_t - \cos\psi_t\tan\sigma_t}{(\sin\psi_t\tan\sigma_t + \cos\alpha\cos\psi_t)\cos(\iota-\theta) - \sin\alpha\sin(\iota-\theta)}$$

Hour angle μ_t

$$\tan\mu_t\Big|_{\iota=\theta} = \frac{\tan\psi_t\cos\sigma_t + \tan\sigma_t}{\tan\psi_t\tan\sigma_t + \cos\alpha}$$

Hour angle μ_t when the gnomon is a style $\iota = \theta$



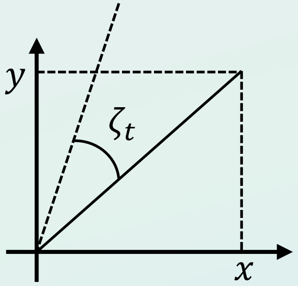
$$\tan\zeta_t = \frac{(\sin i\sin\iota\cos d + \cos i\cos\iota)\tan\mu_t}{1 - \sin^2 i\sin^2 d - (\sin^2 i\cos d\cos\iota - \sin i\sin\iota\cos i)\sin d\tan\mu_t}$$

Shadow angle ζ_t

$$x = \frac{(\cos\psi_t\tan\sigma_t - \sin\psi_t\cos\alpha)\sin d\cos\iota + \Gamma_t\cos d}{\Delta_t}$$

$$y = \frac{(\cos\psi_t\tan\sigma_t - \sin\psi_t\cos\alpha)(\sin i\sin\iota + \cos d\cos i\cos\iota) - \Gamma_t\sin d\cos i}{\Delta_t}$$

Coordinates of the shadow tip x, y



$$\Gamma_t = \sin\psi_t\cos(\iota-\theta)\tan\sigma_t + \cos\alpha\cos\psi_t\cos(\iota-\theta) - \sin\alpha\sin(\iota-\theta)$$

$$\Delta_t = (\cos\psi_t\tan\sigma_t - \sin\psi_t\cos\alpha)\sin d\sin i + (\sin\psi_t\tan\sigma_t + \cos\alpha\cos\psi_t)(\sin i\cos d\cos\theta - \sin\theta\cos i) + (\sin i\sin\theta\cos d + \cos i\cos\theta)\sin\alpha$$

The Solution – Orbit

Earth's orbit as a function of time, and the Equation of Time

$$\psi_t = \rho + \frac{N+1}{N}(t - t_p)$$

$$\sigma_t(\tau) = \frac{b}{a}\frac{\sin(\omega\tau)}{e + \cos(\omega\tau)}$$

$$t(\tau) = a\left(\tau + \frac{e}{\omega}\sin(\omega\tau)\right)$$

ρ is the difference in σ_t between perihelion and winter solstice

N is the number of days in a mean year, approximately 365.2422

t_p is the time of the most recent perihelion

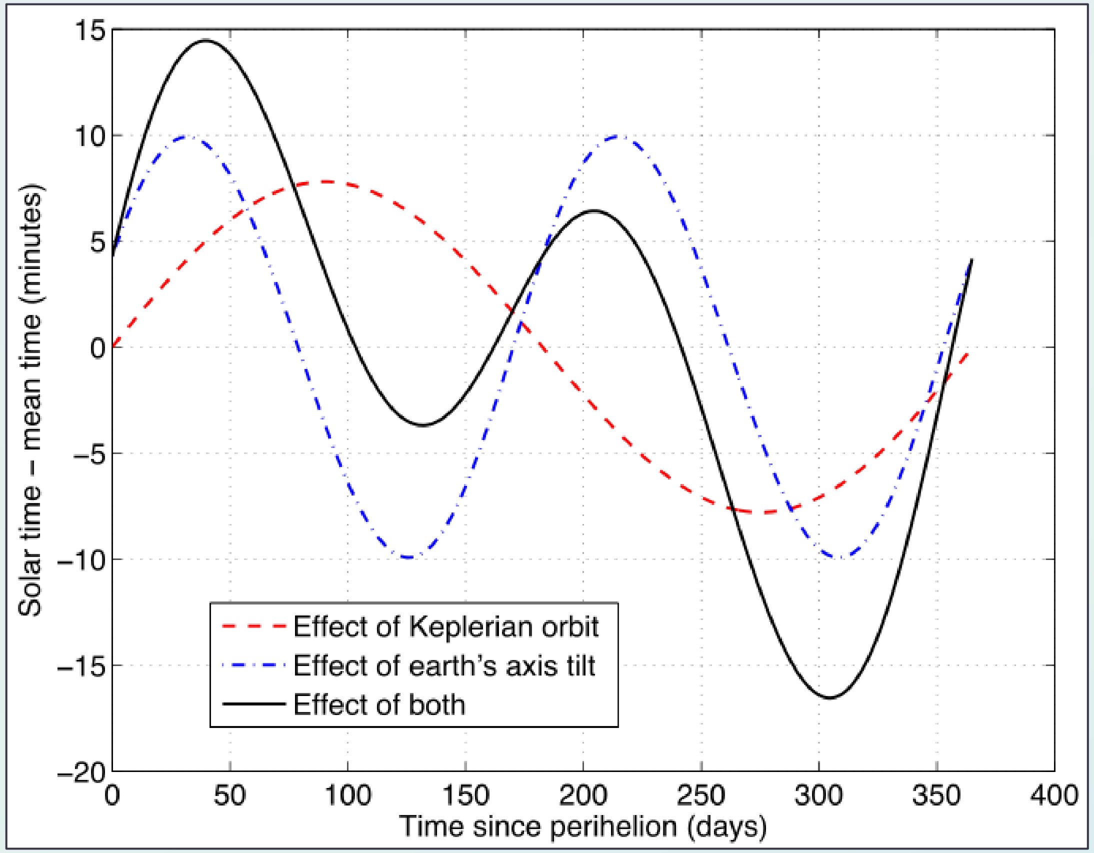
a and b are the semi-major and semi-minor axes of Earth's orbit

e is the eccentricity of Earth's orbit

T_y is length of a year in seconds $(365 \times 24 \times 3600)$

$\omega = \frac{2\pi a}{T_y}$ is an angular frequency parameter in the orbit solution

τ parameterizes the curve traced by Earth's orbit



$$\mu(t) - \mu_{m(t)} = \tan^{-1}\left(\frac{\tan\sigma_t - \tan\psi_t\cos\alpha}{\tan\sigma_t\tan\psi_t + \cos\alpha}\right) - \frac{N+1}{N}\frac{2\pi N}{T_y}(t - t_p)$$

Example Analemmas

Visualizing analemmas for a common dial geometry

The x axis points along the vector m_2 and the y axis points in the direction of $-m_1$.

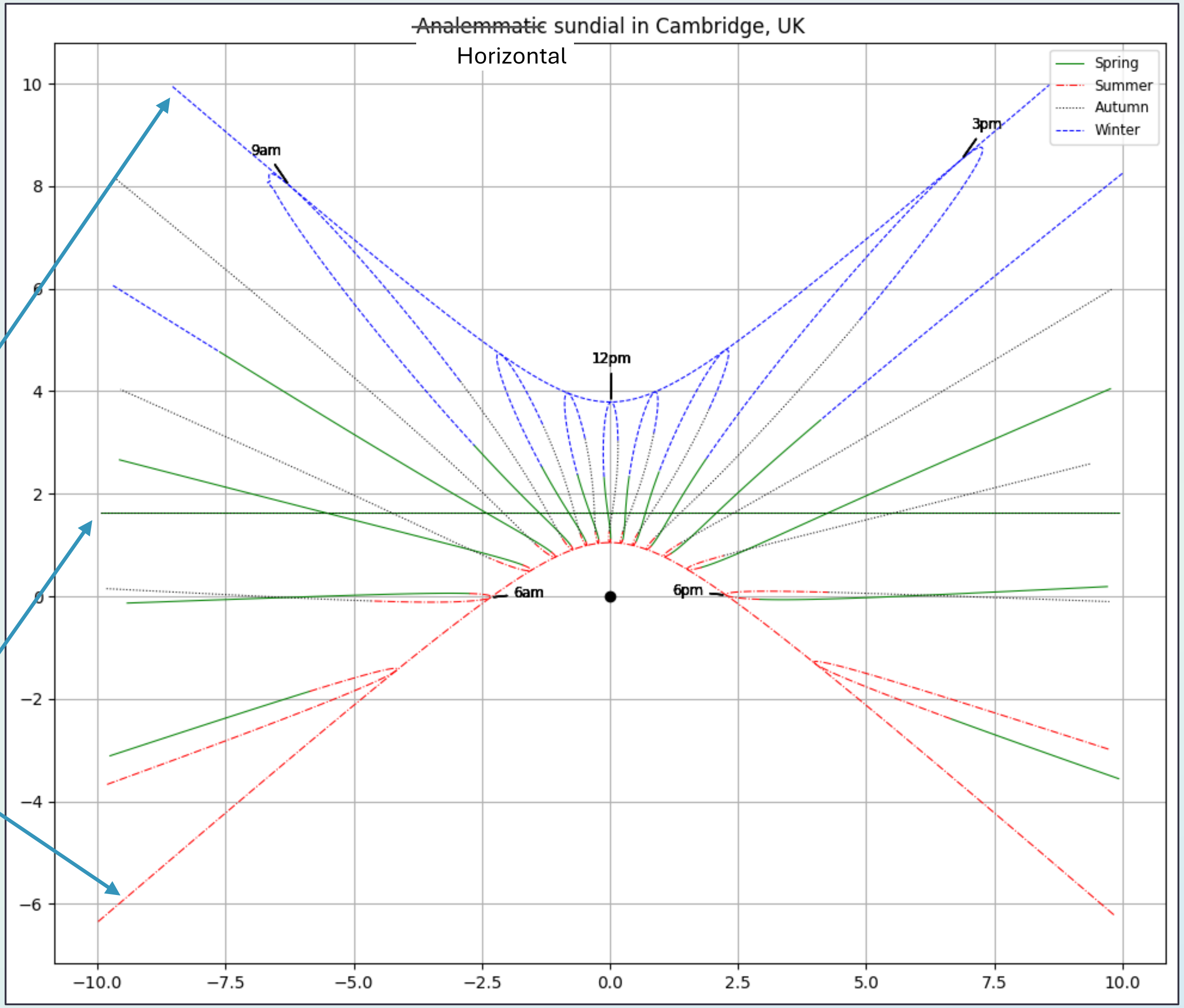
The black dot ● marks the base of the gnomon, and distances are in gnomon lengths.

An analemma for each hour of the day is shown, coloured according to season, with the longest shadows in winter (blue dashed) and the shortest in summer (red dot-dashed).

The path followed by the shadow tip on the solstices and equinoxes is also shown.

On the solstices, the path envelops the analemmas (blue and red).

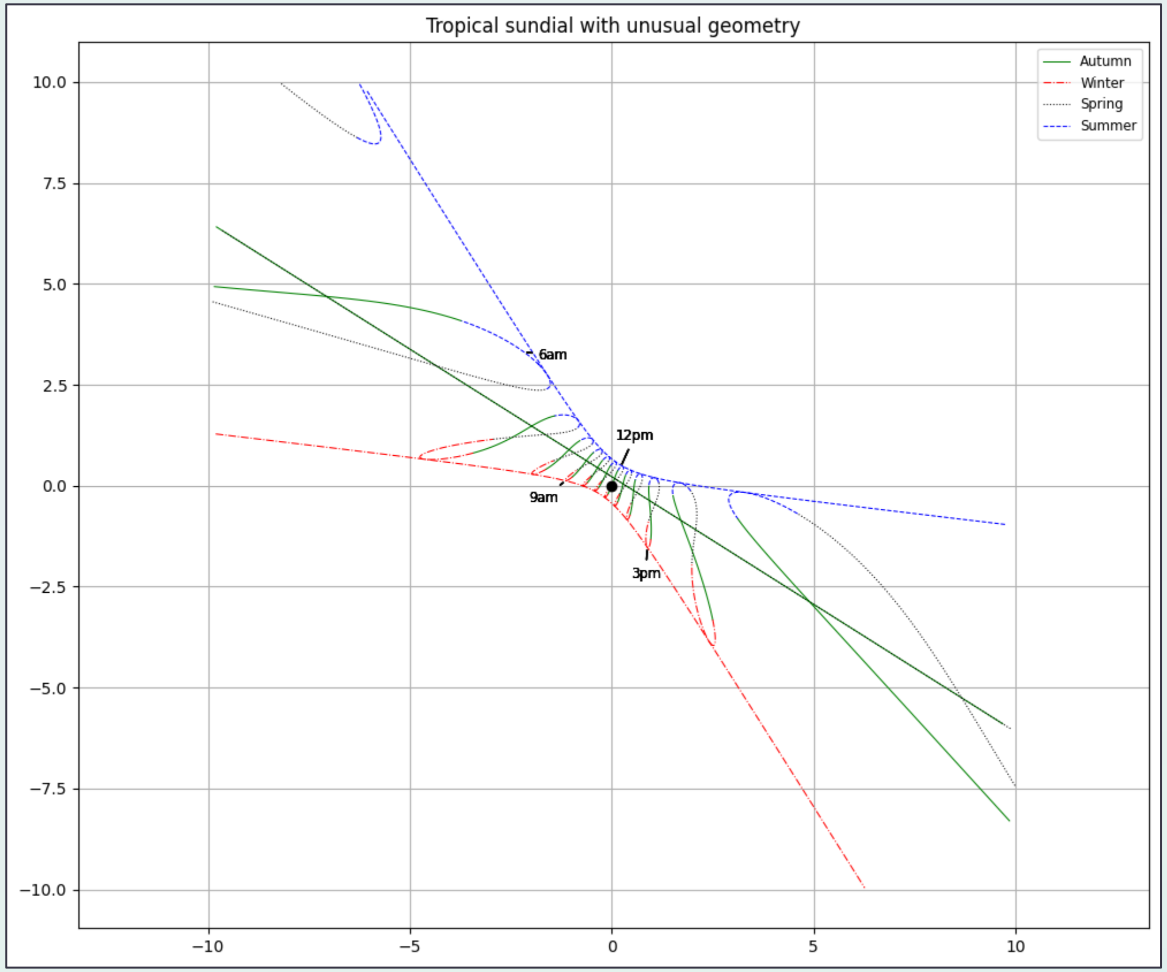
On the equinoxes, this path forms a straight line (green solid and **black** dotted lines on top of each other).



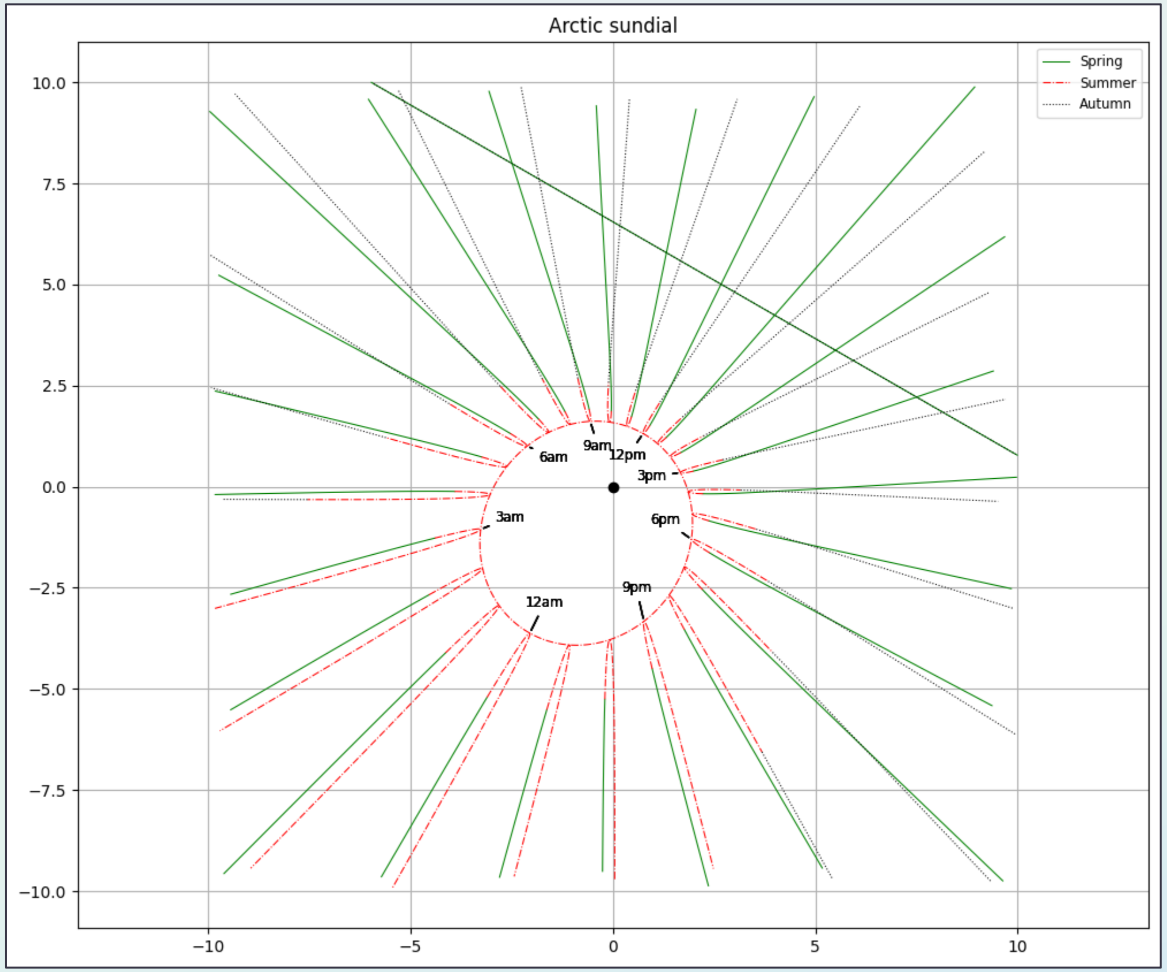
Latitude: $90° - \theta = 52.5°, \iota = 37.5°, i = d = 0$

Example Analemmas

Visualizing analemmas for a couple of unusual sundials



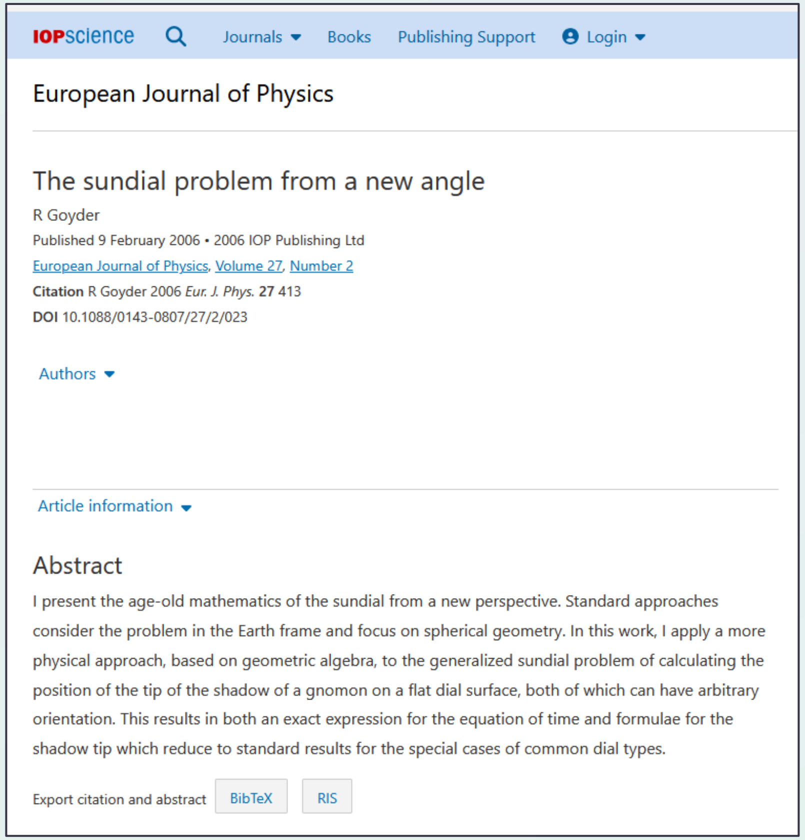
Latitude: $90° - \theta = 100.5°, \iota = 20°, i = -15°, d = 30°$



Latitude: $90° - \theta = 10°, \iota = 0°, i = 0°, d = 30°$

Publication

Available online at
https://russellgoyder.ca/analemma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Authors

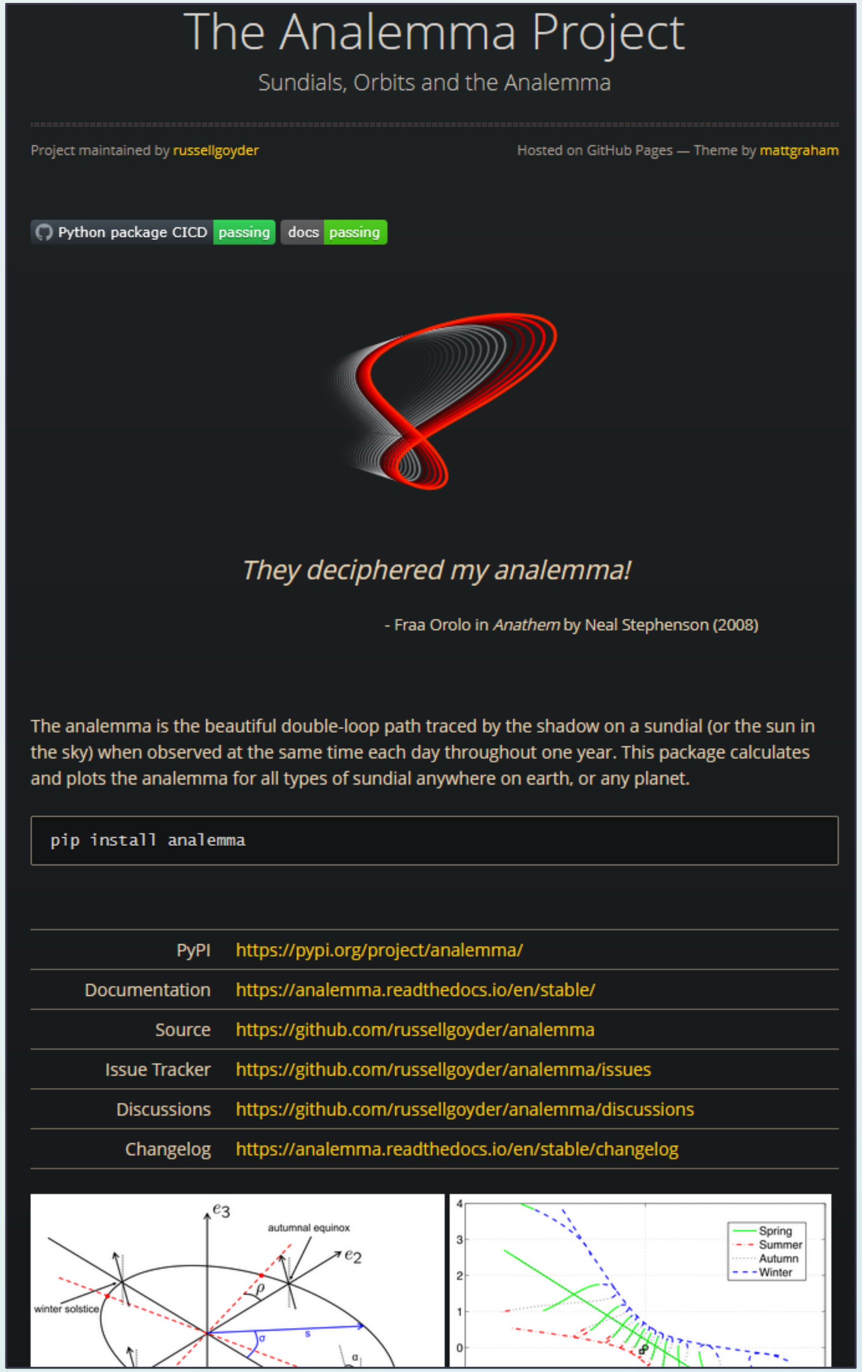
Article information

Abstract

I present the age-old mathematics of the sundial from a new perspective. Standard approaches consider the problem in the Earth frame and focus on spherical geometry. In this work, I apply a more physical approach, based on geometric algebra, to the generalized sundial problem of calculating the position of the tip of the shadow of a gnomon on a flat dial surface, both of which can have arbitrary orientation. This results in both an exact expression for the equation of time and formulae for the shadow tip which reduce to standard results for the special cases of common dial types.

Export citation and abstract BibTeX RIS

Originally published in 2006


The Analemma Project

Sundials, Orbits and the Analemma

Project maintained by russellgoyder Hosted on GitHub Pages — Theme by mattgraham

Python package CICD passing docs passing

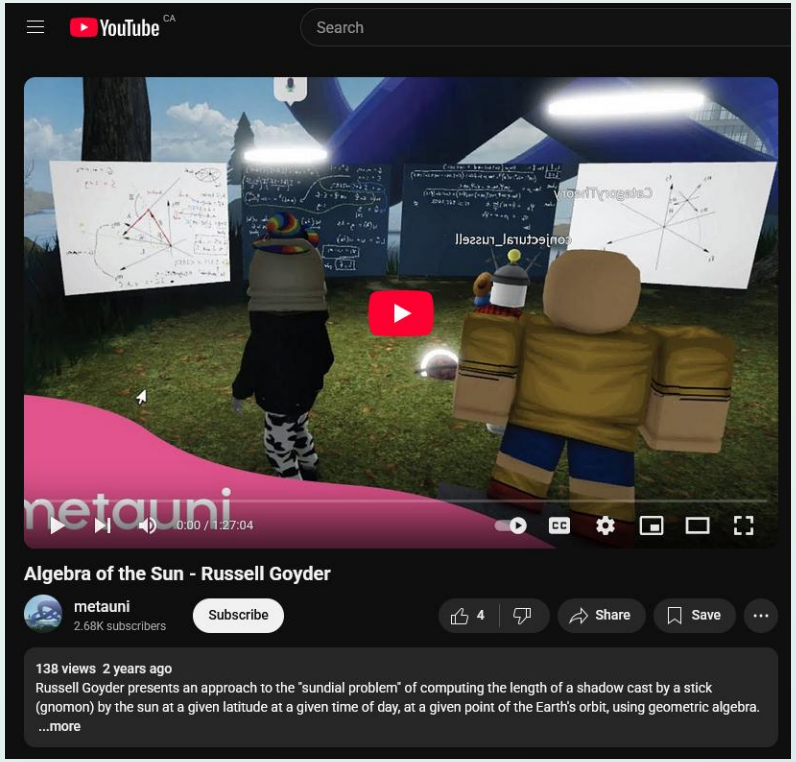
They deciphered my analemma!

- Fraa Orolo in *Anathem* by Neal Stephenson (2008)

The analemma is the beautiful double-loop path traced by the shadow on a sundial (or the sun in the sky) when observed at the same time each day throughout one year. This package calculates and plots the analemma for all types of sundial anywhere on earth, or any planet.

```
pip install analemma
```

PyPI	https://pypi.org/project/analemma/
Documentation	https://analemma.readthedocs.io/en/stable/
Source	https://github.com/russellgoyder/analemma
Issue Tracker	https://github.com/russellgoyder/analemma/issues
Discussions	https://github.com/russellgoyder/analemma/discussions
Changelog	https://analemma.readthedocs.io/en/stable/changelog



Algebra of the Sun - Russell Goyder

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138 views 2 years ago
Russell Goyder presents an approach to the "sundial problem" of computing the length of a shadow cast by a stick (gnomon) by the sun at a given latitude at a given time of day, at a given point of the Earth's orbit, using geometric algebra.
...more

Recently, dusted off to share at metauni.org

Analemma Software

A free Python package that implements and explains the calculation

In the original work I used MATLAB and Maple.

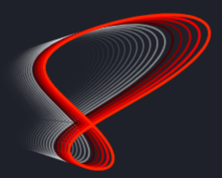
Now, there is a Python version online.

Reproduces every step of the calculation in detail using free tools.

Calculates and plots analemmas and the Equation of Time.

Compares with existing literature (Rohr 1996).

Sundials, Orbits and Analemmas



`analemma` performs sundial calculations, allowing for very general geometry that covers all common types of dial. Highlights include

- exact parametric expressions for the analemma on any type of sundial
- orbits and the equation of time for any planet
- a plotting module to draw the analemma
- all results expressed numerically and in symbolic algebra
- all tests published within the package so you can read and run them

Install

```
pip install analemma
```

Usage

```
import matplotlib.pyplot as plt
from analemma import orbit, plot as aplot, geometry as geom

earth = orbit.PlanetParameters.earth()
vertical_dial = geom.DialParameters.vertical(latitude=52.5) # Cambridge, UK

fig, ax = plt.subplots()
ax.grid()
ax.axis("equal")

aplot.plot_hourly_analemmas(ax, earth, vertical_dial)
```

See Analemma Plots for complete examples showing various analemmas.

For the connection between the angle of the sun, the date, and the time, see The Equation of Time, Sunrise and Sunset, and Orbit Analysis.

Common Dial Types

Sundials are much easier to construct in certain special cases where the gnomon is a parallel to Earth's axis of rotation (known as a style) and the dial face has a special alignment.

The simplest is an equatorial dial, where the dial face is parallel to the equator, and the analemmas are evenly placed around the circle. On the equinoxes, sunrays are parallel to the dial face and so cannot be shown.

A horizontal dial's face is parallel to the ground, while a vertical dial's face is perpendicular. In the northern hemisphere, vertical dial faces south and in the southern hemisphere it faces north.

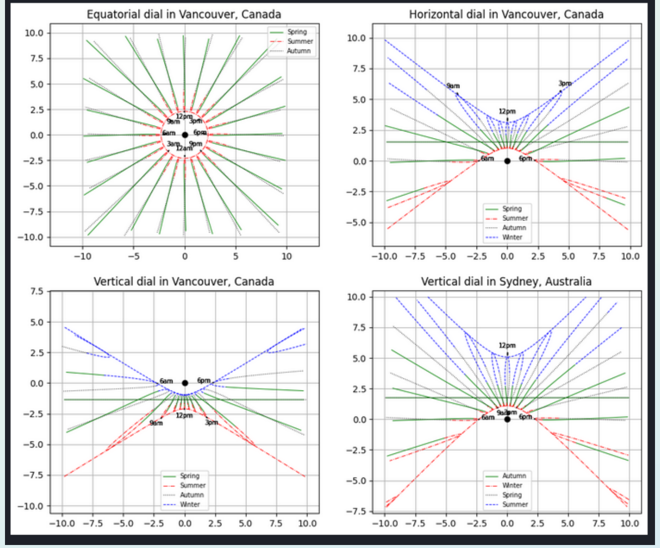
```
fig, axs = plt.subplots(2, 2, figsize=(12, 10))

for ax in axs.flat:
    ax.grid(True)
    ax.axis("equal")

def _analemma_plot(ax, dial, title):
    aplot.plot_hourly_analemmas(ax, earth, dial, year=2024, title=title)

vancouver = 49.3
_analemma_plot(axs[0,0], geom.DialParameters.equatorial(latitude=vancouver), "Equa
_analemma_plot(axs[0,1], geom.DialParameters.horizontal(latitude=vancouver), "Hori
_analemma_plot(axs[1,0], geom.DialParameters.vertical(latitude=vancouver), "Vertic

_analemma_plot(axs[1,1], geom.DialParameters.vertical(latitude=-33.9), "Vertical d
```



Comparison with Rohr's Book

launch binder

The book *SUNDIALS History, Theory and Practice* by Rohr (1996) is the standard reference for sundials and contains many results similar to those present here in `analemma`, except in various special cases, the most general of which is the case where the gnomon is a style, so its inclination $\iota = \theta$, the $(90^\circ$ minus) latitude angle.

In this notebook, demonstrate that our results reduce to those in Rohr when specialized to that case. Throught, we will make use of the following notational translation:

Angle	`analemma`	Rohr
Sundial declination	d	$-d$
Hour angle	μ	HA
Latitude	$90^\circ - \theta$	ϕ
Gnomon-subgnomon angle	A	α
Subgnomon-noon angle	B	β

The function `rohr` below specializes to the case of a style, in the notation of Rohr's book.

```
import sympy as sp
from sympy import sin, cos, tan, cot
from sympy.abc import d, phi, mu, iota, A

from analemma.algebra import frame, util, render, result

def rohr(expr):
    return expr.subs(iota, sp.pi/2-phi).subs(d, -d).simplify()
```

Noon and the Line of Greatest Slope

On page 78 of Rohr's book, he derives the tangent of v, the angle between two lines in the dial face; the noon line and the line of greatest slope. The latter is the vector m_1, the first vector in the dial frame. We can calculate the same as

$$\tan(v) = \frac{\sin(v)}{\cos(v)} = \frac{m_2 \cdot \hat{w}_{\mu=0}}{\ldots}$$

Existing Literature

Seeking to understand whether anything here is novel

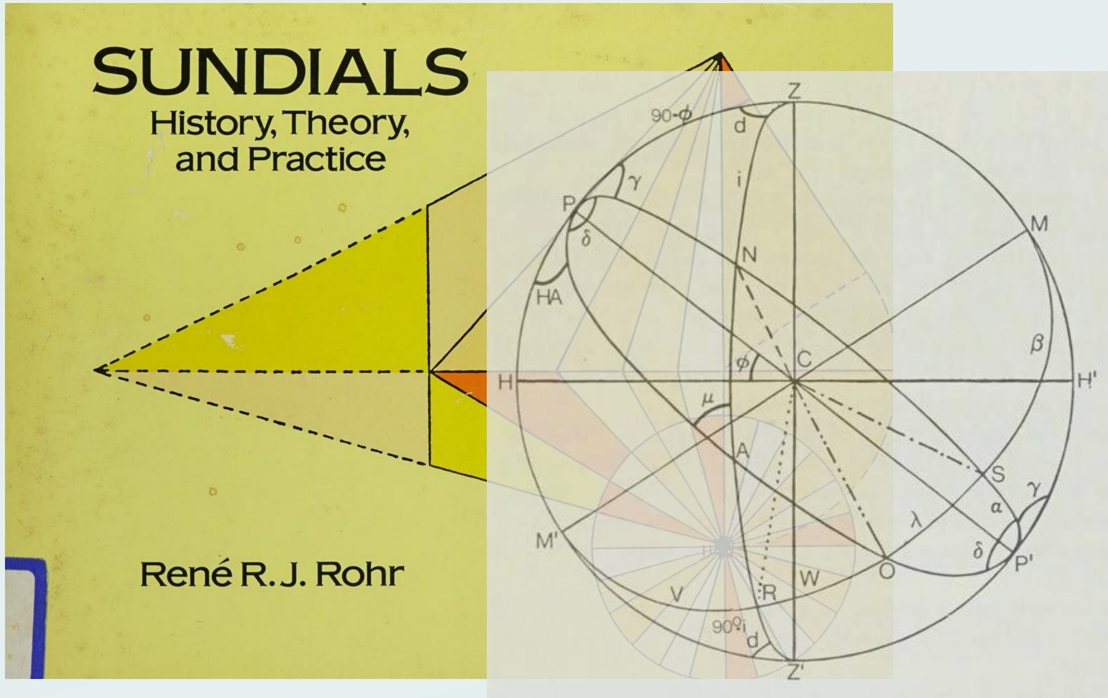


Figure 75, page 76

Although the solution belongs to the field of elementary mathematics, it is not obvious at first view. Yet there are two reasons for describing it. The principal reason is that it leads to general formulae, barely known, giving the elements of any type of dial. The second reason is the hope that it will be welcomed by those who enjoy solving the interesting problems raised by gnomonics and that it may lead to further research.

Page 77

$$\tan w = \frac{\cos i \cos d \sin \phi + \sin i \cos \phi - \cos i \sin d \cot \text{HA}.}{\cos d \cot \text{HA} + \sin d \sin \phi}$$

Page 78

The Shadow and the Line of Greatest Slope

On page 78, Rohr calculates w, the angle between the shadow and the line of greatest slope on the dial face. The equivalent calculation here is $\tan(w) = -y/x$ where x and y are the coordinates of the shadow and the minus sign enters because m_1 points up, not down the line of greatest slope.

```
x, y = result.shadow_coords_xy()
tan_w = (-y/x).subs(cos(mu), cot(mu)*sin(mu))
render.expression(r"\tan(w) = \frac{y}{-x}", rohr(tan_w))
```

$$\tan(w) = \frac{y}{-x} = \frac{-\sin(d)\cos(i)\cot(\mu) + \sin(i)\cos(\phi) + \sin(\phi)\cos(d)\cos(i)}{\sin(d)\sin(\phi) + \cos(d)\cot(\mu)}$$

https://analemma.readthedocs.io/en/stable/nb/rohr_comparison/

What is the value of this work? Is there any beyond fun and pedagogy (see Rohr Fig 75)? In practice, can always proceed numerically.

Rohr's results are special cases (gnomon ⇨ style, $\iota = \theta$). Not obvious to me that they generalize to my results when $\iota \neq \theta$.

I cannot find my approach or results in the literature. It is hard to believe anything here is new given the age of the sundial problem. But maybe?

Thank You

Russell Goyder

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